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DISSIPATIVE DYNAMICS OF SOLITONS IN PLANAR FERROMAGNETS

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Abstract

Dynamics of magnetic bubbles in planar ferromagnets described by the Landau-Lifshitz equation with dissipation is analyzed. The pure $O(3)$ sigma model has static multisoliton solutions, characterized by a number of parameters. The parameters describe a finite dimensional manifold. A small perturbation of energy functional with respect to the sigma model forces solitons to move. Multisoliton dynamics is effectively reduced to a flow in the parameter space.

a. Bogomol'nyi theories. There exists quite a number of classical field theories with the so called Bogomol'nyi[1] energy bound. The classic example of such a theory, which is familiar to both particle physics and condensed matter community, is an $O(3)$ sigma model or a planar ferromagnet[2] described in appropriate units by the energy functional

$$E_{\sigma}[\vec{M}] = \frac{1}{2}\int d^2x \; \partial_k\vec{M}\partial_k\vec{M} \;\; , \tag{1}$$

where $k = 1,2$ runs over planar dimensions and $\vec{M}$ is a 3-component magnetization vector subject to the constraint

$$\vec{M}\;\vec{M} = 1 \;\; . \tag{2}$$

For the energy (1) to be finite the magnetization tends to a constant vector at spatial infinity for any time. For definiteness we take $(0,0,1)$ as this constant vector. With this boundary condition $\vec{M}(t;\vec{x})$ can be viewed as a map from a compactified plane (equivalent to S^2) to the S^2 manifold of magnetization defined by the constraint (2). The energy functional (1) is bounded from below

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$$E_\sigma[\vec{M}] \geq 4\pi|N| \quad . \tag{3}$$

N is an integer topological index of the map $S^2 \to S^2$,

$$N = \int d^2x \; q(t,\vec{x}) \equiv \frac{1}{4\pi}\int d^2x \; \vec{M}[\partial_1\vec{M} \times \partial_2\vec{M}] \quad , \tag{4}$$

where $q(t,\vec{x})$ is a topological charge density. For any winding number N the bound (3) is saturated by a static multisoliton configuration characterized by a finite number of parameters ξ. For example a solution with a negative topological index of $-n$ is given by

$$\begin{aligned} M_B(\vec{x};c,a_i,b_i) &= (\frac{W+\bar{W}}{1+|W|^2}, i\frac{W-\bar{W}}{1+|W|^2}, \frac{1-|W|^2}{1+|W|^2}) \quad , \\ W &= c\frac{(z-a_1)...(z-a_{n-1})}{(z-b_1)...(z-b_n)} \quad . \end{aligned} \tag{5}$$

a's, b's and c are complex parameters, $z = x+iy$. Not all parameters are independent, some different combinations of parameters have to be identified as they give the same $\vec{M}_B$. After this identification there are $4n-1$ real parameters left, they parametrize a $(4n-1)$-dimensional real manifold M_n which will be called a moduli space. It has to be stressed that (5) is a time-independent multisoliton solution, its energy does not depend on the choice of parameters.

b. Relativistic dynamics. The energy functional (1) defines a static version of the sigma model. Dynamics can be introduced to the model in a couple of ways. One dynamical version is a relativistic model described by the Lagrangian

$$L_\sigma[\vec{M}] = \frac{1}{2}\int d^2x \; [\; \partial_t\vec{M}\partial_t\vec{M} - \partial_k\vec{M}\partial_k\vec{M} \;] \quad . \tag{6}$$

A method to study low energy soliton dynamics in a relativistic Bogomol'nyi theory has been proposed by Manton[3] and can be briefly summarized as follows. A Bogomol'nyi solution like (5) saturates the lower energy bound (3). If solitons are forced by initial conditions to move with low velocity, a field configuration at any instant of time remains close to the Bogomol'nyi solution and can be approximated by $\vec{M}_B[\vec{x};\xi(t)]$ with time dependent parameters. The approximation is expected to be the better the lower is velocity. Such an approximate configuration can be substituted to the Lagrangian (6), one obtains after integration over plane an effective low energy Lagrangian

$$\begin{aligned} L_{eff} &= \frac{1}{2}g_{\alpha\beta}(\xi)\dot{\xi}^\alpha\dot{\xi}^\beta \quad , \\ g_{\alpha\beta}(\xi) &= \int d^2x \; \frac{\partial\vec{M}_B}{\partial\xi^\alpha}\frac{\partial\vec{M}_B}{\partial\xi^\beta} \quad . \end{aligned} \tag{7}$$

The field theory (6) is effectively reduced to a finite dimensional mechanical system. The low energy dynamics of solitons is described by a geodesic motion on the moduli space M_n equipped with the metric tensor $g_{\alpha\beta}(\xi)$. The geodesic approximation has been studied in detail for the relativistic sigma model in [4]. It has also been explored in other models like the original BPS theory of monopoles[3] or the abelian Higgs model[5].

c. Landau-Lifshitz equation with dissipation. Another dynamical version of the Bogomol'nyi theory (1), which of main interest for us, is given by

$$\lambda\,\vec{M}\times\partial_t\vec{M}+\hat{\Gamma}(\vec{M})\,\partial_t\vec{M}=\hat{P}_{\vec{M}}[\,\nabla^2\vec{M}+\varepsilon\frac{\delta V}{\delta\vec{M}}(\vec{M})\,]\ \ , \tag{8}$$

where $\hat{P}_{\vec{M}}$ is a projection operator on a subspace orthogonal to $\vec{M}$, defined by $\hat{P}\vec{A}=\vec{A}-\vec{M}(\vec{M}\vec{A})$ for any vector $\vec{A}$. The equation (8) has to be supplemented by the constraint (2). $\hat{\Gamma}(\vec{M})$ is assumed to be a positively definite symmetric matrix for generality. $\varepsilon V[\vec{M}]$ is a small perturbation with respect to the Bogomol'nyi energy (1), the total energy functional is $E=E_\sigma+\varepsilon V$.

Eq. (8) is satisfied for $\varepsilon=0$ by the static multisoliton Bogomol'nyi field (5),

$$\nabla^2\vec{M}_B-\vec{M}_B(\vec{M}_B\nabla^2\vec{M}_B)=0\ \ . \tag{9}$$

For nonzero ε the solitons are no longer static, they move with velocities proportional to ε

$$\dot{\xi}^\alpha=O(\varepsilon)\ \ , \tag{10}$$

where the index α numbers the collective coordinates. At the same time magnetization is given by the Bogomol'nyi field plus a small deviation

$$\vec{M}(t,\vec{x})=\vec{M}_B[\vec{x},\xi(t)]+\varepsilon\,\vec{m}(t,\vec{x})+O(\varepsilon^2)\ \ . \tag{11}$$

The equation (8) does not have Lagrangian formulation in a generic case of $\hat{\Gamma}\neq 0$, one can not proceed along the same lines as in the relativistic case. Instead one has to rely on field equations. Eqs.(10,11) define our perturbative expansion in ε. It follows from the condition (10) that $\partial_t\vec{m}(t,\vec{x})=O(\varepsilon)$. Substitution of Eq.(11) to the field equation (8) and linearization in ε gives

$$\begin{aligned}&\dot{\xi}^\alpha\ [\,\lambda\,\vec{M}_B\times\frac{\partial\vec{M}_B}{\partial\xi^\alpha}+\hat{\Gamma}(\vec{M}_B)\,\frac{\partial\vec{M}_B}{\partial\xi^\alpha}\,]+\varepsilon\,\hat{P}_{\vec{M}_B}\,\frac{\delta V}{\delta\vec{M}}(\vec{M}_B)=\\&\varepsilon\ [\,\nabla^2\vec{m}-\vec{M}_B(\vec{M}_B\nabla^2\vec{m})-(\vec{M}_B\nabla^2\vec{M}_B)\vec{m}-\vec{M}_B(\vec{m}\nabla^2\vec{M}_B)\,]\ \ .\end{aligned} \tag{12}$$

Similar linearization of the constraint (2) leads to a constraint on $\vec{m}$

$$\vec{M}_B\,\vec{m}=0\ \ . \tag{13}$$

$\nabla^2\vec{M}_B$ is parallel to $\vec{M}_B$ according to Eq.(9). Because of this property and the constraint (13) the last term on the RHS of Eq.(12) is zero.

Eq.(12) is a linear inhomogeneous equation for $\vec{m}$. The source term on the LHS of this equation depends on the Bogomol'nyi fields only, the RHS can be interpreted as a linear operator (dependent on $\vec{M}_B[\vec{x},\xi(t)]$) acting on the field $\vec{m}(t,\vec{x})$, say, $\hat{L}\vec{m}$. A projection of Eq.(12) on $\frac{\partial\vec{M}_B}{\partial\xi^\beta}$ somewhat similar as in [6], which is a left zero mode of $\hat{L}$, results in a solvability condition

$$\begin{aligned}&\dot{\xi}^\alpha\ [\,\lambda\,\omega_{\alpha\beta}(\xi)+G_{\alpha\beta}(\xi)\,]-\varepsilon\,F_\beta(\xi)=\\&\varepsilon\int d^2x\,\frac{\partial\vec{M}_B}{\partial\xi^\beta}\ [\,\nabla^2\vec{m}-\vec{M}_B(\vec{M}_B\nabla^2\vec{m})-(\vec{M}_B\nabla^2\vec{M}_B)\vec{m}\,]\ \ ,\end{aligned} \tag{14}$$

where the tensors on the LHS are defined by

$$\begin{aligned}
\omega_{\alpha\beta}(\xi) &= \int d^2x \; \vec{M}_B(\frac{\partial \vec{M}_B}{\partial \xi^\alpha} \times \frac{\partial \vec{M}_B}{\partial \xi^\beta}) \;\; , \\
G_{\alpha\beta}(\xi) &= \int d^2x \; \frac{\partial \vec{M}_B}{\partial \xi^\alpha} \; \hat{\Gamma}(\vec{M}_B) \; \frac{\partial \vec{M}_B}{\partial \xi^\beta} \;\; , \\
F_\beta(\xi) &= -\int d^2x \; \frac{\partial \vec{M}_B}{\partial \xi^\beta} \frac{\delta V}{\delta \vec{M}}(\vec{M}_B) = -\frac{\partial}{\partial \xi^\beta} \int d^2x \; V[\vec{M}_B(\vec{x},\xi)] \;\; .
\end{aligned} \tag{15}$$

If $\hat{\Gamma} = \gamma \hat{1}$ with a constant γ, then $G_{\alpha\beta}(\xi) = \gamma g_{\alpha\beta}(\xi)$ is proportional to the metric tensor (7) on the moduli space, as it was discussed qualitatively in [7]. $F_\beta(\xi)$ can be interpreted as a potential force.

The RHS of Eq.(14) is zero. Clearly the second term of the integrand is zero because $\vec{M}_B \frac{\partial \vec{M}_B}{\partial \xi^\beta} = 0$ thanks to the constraint (2). After integration by parts the RHS of Eq.(14) becomes

$$\varepsilon \int d^2x \; \vec{m} \; [\; \nabla^2 \frac{\partial \vec{M}_B}{\partial \xi^\beta} - \frac{\partial \vec{M}_B}{\partial \xi^\beta}(\vec{M}_B \nabla^2 \vec{M}_B) \;] \;\; . \tag{16}$$

On the other hand taking a derivative $\frac{\partial}{\partial \xi^\beta}$ of Eq.(9) gives

$$\nabla^2 \frac{\partial \vec{M}_B}{\partial \xi^\beta} - \frac{\partial \vec{M}_B}{\partial \xi^\beta}(\vec{M}_B \nabla^2 \vec{M}_B) - \vec{M}_B \frac{\partial}{\partial \xi^\beta}(\vec{M}_B \nabla^2 \vec{M}_B) = 0 \;\; . \tag{17}$$

This equation and the constraint (13) imply that the integral (16) is zero.

To summarize, we have found that a solvability condition for Eqs.(12,13) is given by the following equation of motion for the collective coordinates

$$\dot{\xi}^\alpha \; [\; \lambda \; \omega_{\alpha\beta}(\xi) + G_{\alpha\beta}(\xi) \;] = \varepsilon \; F_\beta(\xi) \;\; . \tag{18}$$

Once again the dynamics of a field theory is reduced to a finite dimensional mechanical system.

d. Example. To substantiate the general discussion by a simple example let us consider the case of one soliton in an external potential and $\hat{\Gamma} = \gamma \hat{1}$ with a constant γ. A general form of one soliton solution is given by $W = \mu/(z-\nu)$, where ν is a complex position of the soliton and the real μ is soliton's size. Nonvanishing tensor elements are

$$\begin{aligned}
\omega_{\nu\bar{\nu}} &= -\omega_{\bar{\nu}\nu} = -2\pi i \;\; , \\
g_{\nu\bar{\nu}} &= g_{\bar{\nu}\nu} = 2\pi \;\; .
\end{aligned} \tag{19}$$

$g_{\mu\mu}$ is divergent on an infinite plane; it follows from the $\beta = \mu$ component of Eq.(18) that μ is constant provided that F_μ is finite. Let the interaction energy with an external potential be given by

$$V = e \int d^2x \; \delta^{(2)}(\vec{x}) \; q(\vec{x},t) \;\; , \tag{20}$$

where q is the topological charge density, which is negative in this case. For $e > 0$ the soliton should be attracted by the impurity at the origin. This form of interaction energy appears for example in a sigma model for quantum Hall ferromagnet[8]. The potential forces are

$$F_\mu = \frac{2e\mu(\nu\bar{\nu} - \mu^2)}{\pi(\nu\bar{\nu} + \mu^2)^3} \ ,$$
$$F_\nu = \bar{F}_{\bar{\nu}} = -\frac{2e\mu^2\bar{\nu}}{\pi(\nu\bar{\nu} + \mu^2)^3} \ . \tag{21}$$

The equation of motion for ν is

$$(\gamma - i\lambda)\dot{\nu} = -(\frac{e}{\pi^2})\frac{\mu^2\nu}{(\nu\bar{\nu} + \mu^2)^3} \ . \tag{22}$$

For $\gamma \neq 0$ the solution of this equation is given by

$$\Theta(t) = \Theta(0) - (\tan B)\ln\frac{R(t)}{R(0)} \ ,$$
$$2\ln\frac{R(t)}{R(0)} + 3[R^2(t) - R^2(0)] + \frac{3}{2}[R^2(t) - R^2(0)]^2 + \frac{1}{3}[R^2(t) - R^2(0)]^3 = -\frac{2\cos B}{A}t \ , \tag{23}$$

where $Ae^{iB} = \pi^2\mu^4(\gamma - i\lambda)/e$, $Re^{i\Theta} = \nu/\mu$ and μ is constant. For $\lambda = 0$ (purely dissipative case) ν relaxes to the equilibrium position at $\nu = 0$ along a radial line. In general it moves towards $\nu = 0$ along spiral lines. In the case of Landau-Lifshitz equation or $\gamma = 0$ the soliton rotates around the origin along an equipotential circular orbit

$$R(t) = R(0) \ ,$$
$$\Theta(t) = \Theta(0) - \frac{t}{A[1 + R^2(0)]^3} \ . \tag{24}$$

e. Conclusion. The central result is the equation (18), which gives a prescription how to deal with dynamics of solitons in dissipative system close to the Bogomol'nyi limit. The equation can be easily generalized to other models because its derivation does not depend much on the special properties of the theory (8).

Generalization to a model with more than one order parameter or field is possible. For such a model it would be natural to expect the relaxation times γ_i for different order parameters to be different, $\gamma_i \neq \gamma_j$ if $i \neq j$. In such a case the tensor $G_{\alpha\beta}$ is not proportional to the metric tensor $g_{\alpha\beta}$ even if γ's are constants.

One of applications could be the dynamics of vortices in superconductors at nonzero temperature. A Bogomol'nyi theory in this case is defined by a Ginzburg-Landau functional for a superconductor at a border between type I and type II superconductivity. An appropriate small perturbation of the quartic potential[9], corresponding to εV, drives the system in the direction of weak type II superconductivity. Somewhat similar free energy functional as for superconductors describes transition from smectic A to nematic phase of liquid crystals[10].

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